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ORIGINAL PAPER

Diffusion driven instability to a drift driven one: Turing patterns in the presence of an electric field

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Abstract We report a general formula for the critical electric field required to trigger a pattern formation in a Turing system in the presence of an electric field (drift term). Our result encompasses all situations from pure diffusion to pure drift.

Keywords Turing system · Reaction-diffusion system · Pattern formation · Electric field · Differential mobility

1 Introduction

Pattern formation in two reacting and diffusing systems was first studied by Turing [1,2]. Turing argued that if the diffusion coefficients of the two species are widely different, then if one of the species is auto-catalytic with the other inhibiting its growth, then the steady homogenous state will be unstable to a patterned steady state. The instability could also set in a temporal pattern in a spatially homogeneous state under certain conditions. Turing patterns have been a very important aspect of the study of non-linear systems [3–10]. Decades later it was found by Rovinsky and Menzinger

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[11–13] that pattern formation could occur even if the two diffusion coefficients were nearly equal provided there was an external electric field and the diffusion species were charged giving rise to a gradient coupling. The pattern formed in this case would be drift induced travelling waves as opposed to the stationary patterns of Turing. Recently there has been extension of the Turing work in some unexpected directions [14–16]. In a recent study, Riaz et al. [17,18] have shown numerically that a Turing pattern for charged species could be altered by an applied electric field [19]. We will demonstrate here that some analytic results can be derived that help understand the numerical results of Riaz et al. [17].

In this paper, we revisit the pattern formation problem with an external electric field to arrive at a single analytic result that has the property of capturing all the possible conditions for the instabilities in the system. It should be noted that while the work described here refers to chemical pattern formation, the analysis should also generalise in a situation where the species are neutral but kept in a medium which has a velocity in a definite direction. This should be a case of an advective field. Here we work with a two dimensional set up as in the work of Riaz et al. We take the system to be unbounded in the y-direction and bounded by walls in the x-direction at x = ±L. The boundary conditions are that the concentration of the species vanishes at the boundary and so does the current normal to the plates which is proportional to the x-direction of the concentrations. The electric field is taken to be in the x-direction which leads to a drift in that direction. The existence of the plates in the x-direction is vital to keep the system bounded. The plates also play the very important role of fixing the wave number in the x-direction. In the absence of the constraint there will be an overall selection mechanism for the wave numbers ($k = k_1 + k_2$) but the individual components are not uniquely determined. What we will se below is that k_1 is fixed by the boundary condition and thus once k^2 is known both k_1 and k_2 will be determined. In the work of Riaz et al. [17] the fixing of k_1 and k_2 is undertaken numerically. Here the analytic fixing of k_1 allows us to find a general expression for the thresholds of the different instabilities. The central result of our paper is expressed in Eq. (23) which includes stability results for all possible situations.

2 Reaction-diffusion system with electric field effects

The general reaction-diffusion problem for two species A(x,y,t) and B(x,y,t) can be modelled by the evolution equations [17]

$$\frac{\partial A}{\partial t} = D\nabla^2 A + z_2 ED\frac{\partial A}{\partial x} + f(A, B) \tag{1}$$

$$\frac{\partial B}{\partial t} = \nabla^2 B + z_1 E\frac{\partial B}{\partial x} + \mathrm{g(A, B)} \tag{2}$$

In the above D is the diffusion coefficient for the species A in units where the diffusion coefficient for the species B is unity. The external electric field in the x-direction is denoted by E and z_1 and z_2 are the charges associated with the inhibitor and activator respectively. The operator ∇^2 is two dimensional (pattern on a substrate) and the function f(A,B) and g(A,B) describe the growth and interaction of the species A and B. The electric field terms come from an expression for the current together with relevant Einstein's relation. In the Gierer–Meinhardt model [20,21].

$$\mathrm{f(A,B)} = \frac{A^2}{B} - \mathrm{A} + \sigma$$
$$\mathrm{g(A,B)} = \mu(\mathrm{A}^2 - \mathrm{B})$$

The growth rate of A due to interaction with the substrate is σ and the natural decay rate for B is μ. In the Lengyel–Epstein model [22].

$$\mathrm{f(A,B)} = \sigma \mathrm{b}\left(\mathrm{B} - \frac{AB}{1+B^2}\right)$$
$$\mathrm{g(A,B)} = \mathrm{a} - \mathrm{B} - \frac{4AB}{1+B^2}$$

where σ, b and a are constants. The homogenous steady state is $\mathrm{A} = \mathrm{A}_o$ and $\mathrm{B} = \mathrm{B}_o$ such that $\mathrm{f}(\mathrm{A}_o, \mathrm{B}_o) = \mathrm{g}(\mathrm{A}_o, \mathrm{B}_o) = 0$. The linear stability analysis around $\mathrm{A} = \mathrm{A}_o$ and $\mathrm{B} = \mathrm{B}_o$ leads to

$$\frac{d(\delta A)}{dt} = \mathrm{D}\nabla^2\delta A + \mathrm{z}_2\mathrm{ED}\frac{\partial(\delta A)}{\partial x} + \mathrm{a}_{11}\delta A + \mathrm{a}_{12}\delta B \tag{3}$$
$$\frac{d(\delta B)}{dt} = \nabla^2\delta B + \mathrm{z}_1\mathrm{E}\frac{\partial(\delta B)}{\partial x} + \mathrm{a}_{21}\delta A + \mathrm{a}_{22}\delta B \tag{4}$$

where $\mathrm{a}_{11} = \left(\frac{\partial f}{\partial A}\right)_{A_oB_o}$, $\mathrm{a}_{12} = \left(\frac{\partial f}{\partial B}\right)_{A_oB_o}$, $\mathrm{a}_{21} = \left(\frac{\partial g}{\partial A}\right)_{A_oB_o}$ and $\mathrm{a}_{22} = \left(\frac{\partial g}{\partial B}\right)_{A_oB_o}$.

We consider a geometry which is confined by plates at x = ±L and is unbounded in the y-direction. The solution will be periodic in y-direction and if we take the wave number in this direction to be k_2, then we can write

$$(\delta\mathrm{A}, \delta\mathrm{B}) = (\mathrm{A}_1(\mathrm{x}), \mathrm{B}_1(\mathrm{x}))\mathrm{e}^{(ik_2y)}\mathrm{e}^{(\lambda t)} \tag{5}$$

where λ is the eigen value determining the growth. Then $\mathrm{A}_1(\mathrm{x})$ and $\mathrm{B}_1(\mathrm{x})$ satisfy the differential equation

$$\left[\lambda + Dk_2^2 - D\frac{d^2}{dx^2} - z_2ED\frac{d}{dx} - a_{11}\right]\mathrm{A}_1 = \mathrm{a}_{12}\mathrm{B}_1 \tag{6}$$
$$\left[\lambda + k_2^2 - \frac{d^2}{dx^2} - z_1E\frac{d}{dx} - a_{22}\right]\mathrm{B}_1 = \mathrm{a}_{21}\mathrm{A}_1 \tag{7}$$

Eliminating B_1 one can write

$$\mathrm{D}\frac{d^4A_1}{dx^4}+\bar{\alpha_2}\frac{d^3A_1}{dx^3}+[(\mathrm{a}_{11})+\mathrm{Da}_{22}-2\mathrm{D}_2^2-\lambda(1+\mathrm{D})+\mathrm{z}_1\mathrm{z}_2\mathrm{DE}^2]\frac{d^2A_1}{dx^2}$$
$$+[\bar{\beta_2}-\lambda(z_1+z_2D)E]\frac{dA_1}{dx}+[\triangle+\lambda^2-\lambda\bar{\alpha_1}-k_2^2(a_{11}+Da_{22})+Dk_2^4]\mathrm{A}_1=0 \tag{8}$$

where

$$\begin{aligned}\bar{\alpha_1}&=a_{11}+a_{22}-(1+D)k_2^2\\ \bar{\alpha_2}&=ED(z_1+z_2)\\ \bar{\beta_2}&=\mathrm{E}[z_1a_{11}+z_2Da_{22}-(z_1+z_2)Dk_2^2]\\ \triangle&=a_{11}a_{22}-a_{12}a_{21}\end{aligned}$$

At this point the general procedure should be clear. We need to solve the homogenous fourth order equation above. This will involve four arbitrary constants which have to be fixed by boundary conditions. Since the system is homogenous the four conditions will lead to four homogenous linear algebraic equations and for consistency the determinant has to vanish. The resulting equation fixes λ in terms of L, k_2, E and other system parameters. The requirement $\mathrm{Re}\lambda \geq 0$ for instability allows us to discuss the different situation that can occur. The above procedure is general and in principle cumbersome. We illustrate this in the simpler situation of E = 0 and the Turing limit i.e. $D \ll 1$. The lesson that we learn here will be put to good use for the complicated case.

3 Wave number selection for E = 0

We proceed by setting E = 0 in Eq. (8) and obtain

$$\mathrm{D}\frac{d^4A_1}{dx^4}+[a_{11}+Da_{22}-2Dk_2^2-\lambda(1+D)]\frac{d^2A_1}{dx^2}+[\lambda^2-\bar{\alpha_1}\lambda+Dk_2^4$$
$$-k_2^2(a_{11}+Da_{22})+\triangle]\mathrm{A}_1=0 \tag{9}$$

We assume a trial solution

$$\mathrm{A}_1=\mathrm{ce}^{(imx)} \tag{10}$$

where $i=\sqrt{-1}$. Then from Eq. (9) we have

$$\mathrm{Dm}^4-[\mathrm{Da}_{22}+\mathrm{a}_{11}-2\mathrm{Dk}_2^2-\lambda(1+D)]\mathrm{m}^2+[\lambda^2-\bar{\alpha_1}\lambda+Dk_2^4$$
$$-k_2^2(a_{11}+Da_{22})+\triangle]=0 \tag{11}$$

For $D \ll 1$ (The Turing case) the two roots are approximately (we keep Dk_2^2 since k_2 is not known a-priori)

$$m_1^2 \simeq \frac{a_{11} - \lambda - 2Dk_2^2}{D} \tag{12}$$

$$m_2^2 \simeq \frac{\lambda^2 - \alpha\lambda - k_2^2(a_{11} + Da_{22}) + Dk_2^4 + \triangle}{a_{11} - \lambda - 2Dk_2^2} \tag{13}$$

where $\alpha = a_{11} + a_{22} - k_2^2$.

For even solutions, we can write

$$A_1 = c_1\cos(m_1 x) + c_2\cos(m_2 x) \tag{14}$$

We impose the boundary condition that the fluxes $\frac{dA_1}{dx}$ and $\frac{dB_1}{dx}$ vanishes at $x = \pm L$. Now if we use Eq. (6) with $E = 0$ and take derivatives with respect to x, it is immediately clear that $\frac{d^3A_1}{dx^3}$ has to vanish at $x = \pm L$.

In the Turing limit $m_1 << m_2$ and hence $\cos(m_1 x)$ will have a fast oscillation which will average out to zero. This forces the wave number selection $m_2 L = \pi$ and Eq. (11) becomes with $m = m_2 = \frac{\pi}{L}$

$$\lambda^2 - \left[TrA - (1+D)\left(k_2^2 + \frac{\pi^2}{L^2}\right)\right]\lambda - \left(k_2^2 + \frac{\pi^2}{L^2}\right)a_{11} - Da_{22}\frac{\pi^2}{L^2} + D\left(\frac{\pi^2}{L^2}\right)^2$$
$$+ \triangle + Dk_2^4 - Da_{22}k_2^2 + 2Dk_2^2\frac{\pi^2}{L^2} = 0 \tag{15}$$

where $TrA = a_{11} + a_{22}$.

We note that $k_2^2 + \frac{\pi^2}{L^2}$ enters as a combination of which we call k^2. In that case Eq. (15) becomes

$$\lambda^2 - (TrA - (1+D)k^2)\lambda - (a_{11} + Da_{22})k^2 + Dk^4 + \triangle = 0 \tag{16}$$

This reproduces the Turing condition to the leading order in D, since we find the condition for instability is

$$\triangle - \frac{(a_{11} + Da_{22})^2}{4D} < 0 \tag{17}$$

and the characteristic wave number k_{min} is given by

$$k_{min}^2 = \frac{(a_{11} + Da_{22})}{2D} \tag{18}$$

4 Instability condition for E $\neq 0$

The lesson that we learnt from the above exercise is that the operator $\frac{d}{dx}$ can be effectively replaced by $i\frac{\pi}{2L}$ and using Eq. (8), we determine λ from

$$\lambda^2 - \lambda[TrA - (1+D)k^2 + i\frac{\pi}{2L}(z_1 + z_2 D)] + \mathrm{D}k^4 - \mathrm{k}^2(\mathrm{a}_{11} + \mathrm{Da}_{22})$$
$$+ \triangle - \frac{i\pi}{2L}ED(k^2 - a_{22}) - \frac{i\pi}{2L}Ez_1(Dk^2 - a_{11}) - \frac{\pi^2}{4L^2}z_1 z_2 E^2 D = 0 \quad (19)$$

The above equation can be written as

$$\lambda^2 - \lambda(\alpha_1 + i\alpha_2) + \beta_1 + i\beta_2 = 0 \quad (20)$$

where

$$\begin{aligned}
\alpha_1 &= TrA - (1+D)k^2 \\
\alpha_2 &= -\frac{\pi}{2L}E(z_1 + z_2 D) \\
\beta_1 &= Dk^4 - k^2(a_{11} + Da_{22}) + \triangle - \frac{\pi^2}{4L^2}E^2 z_1 z_2 D \\
\beta_2 &= -\frac{\pi}{2L}E[z_1 a_{11} + z_2 Da_{22} - (z_1 + z_2)Dk^2]
\end{aligned} \quad (21)$$

The real part of the eigen value λ in Eq. (19) will be negative (the condition that the homogenous state will be stable) provided

$$\beta_2^2 < \alpha_1(\alpha_1\beta_1 + \alpha_2\beta_2) \quad (22)$$

using $\alpha_1, \alpha_2, \beta_1, \beta_2$ given in Eq. (21) leads after straightforward algebra to the central result

$$(z_1 - Dz_2)^2[a_{11}a_{22} - k^2(a_{11} + Da_{22}) + Dk^4]\frac{\pi^2}{4L^2}E^2$$
$$> [TrA - (1+D)k^2]^2[k^2(a_{11} + Da_{22}) - \triangle - Dk^4] \quad (23)$$

This is our primary result. This is the single formula that contains all the possibilities of the pattern formation in the presence of electric field.

We will now explore the various possibilities.

First consider the Turing problem E = 0. Now the right hand side of Eq. (23) has to be negative for stability which means

$$\triangle - k^2(a_{11} + Da_{22}) + Dk^4 > 0 \quad (24)$$

for stability. This is the central criterion for stability for all $\triangle > 0$ in the presence of diffusion. If the sign is reversed in Eq. (24), we get the Turing instability and for

$\triangle > 0$, this occurs for a band of wave number where the above expression is negative. The minimum of the expression is obtained for $k^2 = \frac{a_{11}+Da_{22}}{2D}$ and the value at the minimum is $\triangle - \frac{(a_{11}+Da_{22})^2}{4D}$ and hence the instability criterion is $\triangle < \frac{(a_{11}+Da_{22})^2}{4D}$, an inequality which easy to satisfy for $D \ll 1$.

We now consider the opposite limit i.e. there is no diffusion and only drift. In this case Eq. (23)

$$(z_1 - Dz_2)^2 a_{11}a_{22}\frac{\pi^2}{4L^2}E^2 > (TrA)^2(-\triangle) \tag{25}$$

Since we want to start with an initially stable state i.e. TrA<0 and $\triangle 0 >$, we have $a_{11}a_{22}$ <0 and Eq. (25) becomes

$$\frac{\pi E}{2L} < \frac{|TrA|}{|z_1 - Dz_2|}\left(\frac{-\triangle}{a_{11}a_{22}}^{1/2}\right) \tag{26}$$

This clearly shows that instability will set in if $E > E_o$ where

$$\frac{\pi E_o}{2L} = \frac{|TrA|}{|z_1 - Dz_2|}\left(\frac{-\triangle}{a_{11}a_{22}}^{1/2}\right) \tag{27}$$

We see immediately that for the instability to set in one must have a differential mobility i.e. $z_1 \neq Dz_2$. This result is in accordance with Rovinsky and Menzinger [9]. In the case of $D \simeq 1$ i.e. the two diffusivities are nearly equal (a situation very different from Turing), we get for instability

$$\begin{aligned}(z_1 - z_2)^2[a_{11}a_{22} - k^2(a_{11} + a_{22}) + Dk^4]\frac{\pi^2}{4L^2}E^2 \\ > [TrA - 2k^2]^2[k^2(a_{11} + a_{22}) - \triangle - k^4]\end{aligned} \tag{28}$$

We treat the situation which for E = 0 is stable so far as the reaction goes and is also stable when diffusion is included. This implies $\triangle > 0$, TrA < 0, $\triangle - (a_{11} + a_{22})k^2 + k^4 > 0$. The right hand side of Eq. (28) is now negative and for the inequality to hold, we need $a_{11}a_{22} - k^2(a_{11} + a_{22}) + k^4 < 0$. With $a_{11}a_{22} < 0$ and TrA < 0, we can satisfy the inequality in the range $0 \leq k \leq k_o$ where

$$2k_o^2 = \mathrm{TrA} + \sqrt{Tr(A) - 2a_{11}a_{22}} \tag{29}$$

and for $k < k_o$, the critical value of E which will trigger an instability will be given by

$$\frac{\pi^2}{4L^2}E^2 > \frac{(TrA - 2k^2)^2}{(z_1 - z_2)^2}\frac{k^2(a_{11} + a_{22}) - \triangle - k^4}{a_{11}a_{22} - k^2(a_{11} + a_{22}) + k^4} \tag{30}$$

According to the above relation all wave numbers greater than k_o are always stable. In writing down the above condition, we see the advantage of the exact expression of Eq. (23). The order of magnitude estimation in [9] does not yield the above answer.

It is interesting to note that there is a clear demarkation of the roles of the diffusion and electric field in the wave number selection of the instability. If the diffusivities of the two species A and B are widely different, then the fast diffusion of the species B which is antagonistic to the species A Eqs. (1) and (2) ensures that the influence of the local fluctuation of A is restricted to a small region of space which means that the ensuing pattern would have a small wavelength and a large k value. In this range the rapid spatial variation makes the constant electric field ineffective. This probably indicates that a spatially varying electric field with the correctly chosen wavelength may have a significant impact for $D \ll 1$. When the diffusion rate of both the species are of the same order, then the Turing pattern would have a larger wavelength and now the electric field can have a non negligible impact. That is why the instability triggered by the electric field always has a wavelength larger than a critical value. For $D \ll 1$, we expect that the constant electric field will have little effect.

At this point we would like to compare with the findings of [17]. We use the same parameters as they did namely $a_{11} = 0.899$, $a_{12} = 1$, $a_{21} = -0.899$ and $a_{22} = -0.91$. This ensures $\mathrm{Tr}\, A < 0$, $|A|(\triangle) > 0$ and also that there is no diffusion induced instability. In [17], the authors have taken two values of E which they numerically find to be above some threshold so that over a restricted range of small wave numbers the instability against a patterned state can be seen. From Eq. (30). we can actually read off the threshold above which the electric field needs to be set and also note that the wave number has to be restricted to be below a cutoff given by Eq. (29).The relevant plot for the threshold is shown in Fig. 1. Not only is this completely consistent with [17] but gives the threshold for E for onset of instability as a function of the wave number

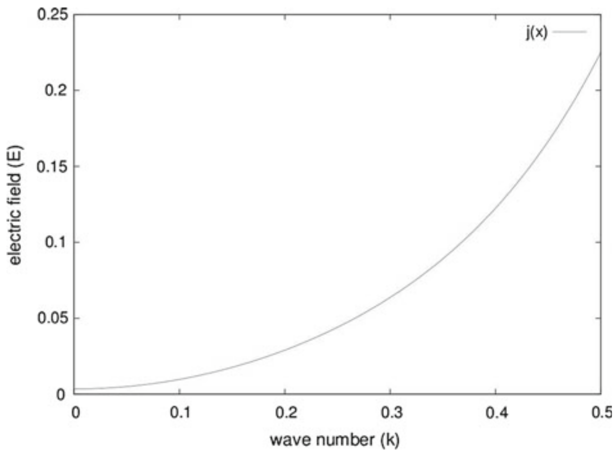

Fig. 1 Plot of critical electric field (E) versus wave number (k) for $a_{11} = 0.899, a_{22} = -0.91, a_{12} = 1, a_{21} = -0.899, z_1 = 1$ and $z_2 = 2$

for any values of the system parameters. For an arbitrary diffusion coefficient D, our basic results can be stated thus

- If for any D, diffusion destabilizes a stable reactive system, then $k^2(a_{11}+Da_{22})-\triangle-Dk^4>0$ and it follows that $a_{11}a_{22}-k^2(a_{11}+Da_{22})+Dk^4=a_{11}a_{22}-\triangle+[\triangle-k^2(a_{11}+Da_{22})+Dk^4]<0$ since $a_{11}a_{22}<0$. Hence Eq. (23) can never be satisfied. No amount of electric field can stabilize the system.
- If the diffusive system is stable i.e. $k^2(a_{11}+Da_{22})-\triangle-Dk^4<0$, then a critical $E(E_c)$ will destabilize the state provided $a_{11}a_{22}-k^2(a_{11}+Da_{22})+Dk^4<0$ which will happen if $k<k_o'$ given by

$$k_o^{2'}=\frac{a_{11}+Da_{22}+\sqrt{(a_{11}+Da_{22})^2-4Da_{11}a_{22}}}{2D} \tag{31}$$

For small enough D, $k_o^{2'}\simeq\frac{a_{11}+Da_{22}}{D}$ and

$$\frac{\pi^2}{4L^2}E_c^2 \simeq \frac{[TrA-(1+D)k^2]^2}{(z_1-Dz_2)^2}\left(\frac{-\triangle}{a_{11}a_{22}}\right) \tag{32}$$

5 Conclusion

In summary we have studied a reaction-diffusion system in the presence of a constant electric field along a particular direction. We have found a single analytic expression which contains all possible information about the stability and the instability of the system for different ranges of the diffusion coefficient. The primary result that emerges is that there is an upper limit on the wave number of the instability and that for each wave number below that there is a critical electric field that can excite that particular wave number.

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